Caringbah High School

Year 12 2021
Mathematics Extension 1
HSC Course
Assessment Task 4

General Instructions
- Reading time – 10 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A reference sheet is provided
- In Questions 11-14, show relevant mathematical reasoning and/or calculations
- Marks may not be awarded for partial or incomplete answers

Total marks – 70

Section I **10 marks**
Attempt Questions 1-10
Mark your answers on the answer sheet provided. You may detach the sheet and write your name on it.

Section II **60 marks**
Attempt Questions 11-14
Write your answers in the answer booklets provided. Ensure your name or student number is clearly visible.

Name: ______________________________ **Class:** __________

Marker's Use Only						
Section I	**Section II**				**Total**	
Q 1-10	**Q11**	**Q12**	**Q13**	**Q14**		
/10	/15	/15	/15	/15	/70	%

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10

1. Which of the following is the exact value of $\displaystyle\int_{\frac{3}{\sqrt{2}}}^{3} \frac{4dx}{\sqrt{9-x^2}}$

(A) $-\pi$

(B) $-\dfrac{\pi}{4}$

(C) $\dfrac{\pi}{4}$

(D) π

2. An oil slick is in the shape of a circle. Its surface area is increasing at a rate of $10\,\text{m}^2/\text{s}$. Let *r* metres be the radius of the oil slick after *t* seconds.
The rate of increase of *r,* in m/s, is given by

(A) $\dfrac{5}{\pi r}$

(B) $\dfrac{20}{\pi r}$

(C) $\dfrac{10}{\pi r^2}$

(D) $\dfrac{1}{20\pi r}$

3. The equation $x^3 - 2x^2 - 4x + 8 = 0$ has roots α, β and γ.

What is the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$?

(A) $-\frac{1}{2}$

(B) $-\frac{1}{4}$

(C) $\frac{1}{4}$

`

(D) $\frac{1}{2}$

4. Given that α and β are both acute angles, evaluate $\sin(\alpha + \beta)$ if $\sin\alpha = \frac{8}{17}$ and $\sin\beta = \frac{4}{5}$

(A) $\frac{108}{85}$

(B) $\frac{84}{85}$

(C) $\frac{36}{85}$

(D) $\frac{28}{85}$

5. Given that $y = \cos^{-1}\left(\frac{1}{x}\right)$, the correct expression for $\frac{dy}{dx}$ is:

(A) $\frac{1}{\sqrt{x^2 - 1}}$

(B) $\frac{1}{x\sqrt{x^2 - 1}}$

(C) $\frac{-1}{\sqrt{x^2 - 1}}$

(D) $\frac{-1}{x\sqrt{x^2 - 1}}$

6. Find $\int \sec^2\theta \tan^2\theta \, d\theta$

(A) $\sec^2\theta + \frac{1}{2}\tan^2\theta + C$

(B) $\frac{1}{3}\tan^3\theta + C$

(C) $\frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta + C$

(D) $\tan^4\theta - \ln\left|\cos^4\theta\right| + C$

7. A committee of 5 people is to be chosen from a group of 6 girls and 4 boys. How many different committees could be formed that have a least one boy?

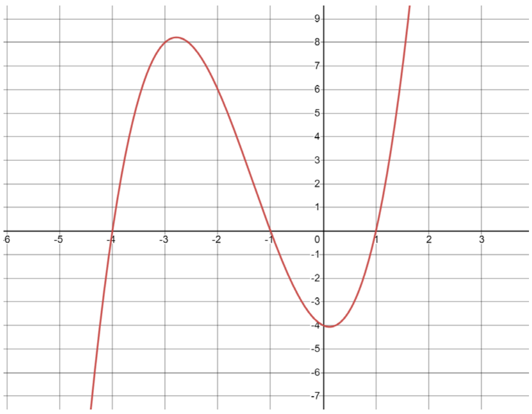
(A) $\binom{10}{5} - 1$

(B) $\binom{4}{1} \times \binom{6}{4}$

(C) $\binom{4}{1} + \binom{6}{4}$

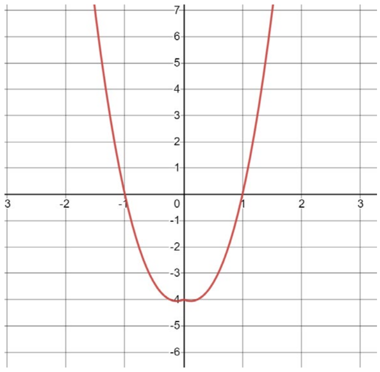
(D) $\binom{10}{5} - 6$

8.

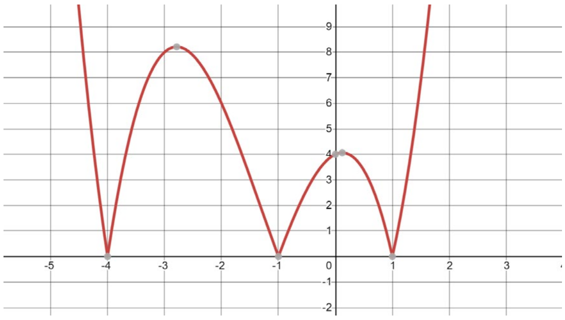


The graph of $y = f(x)$ is shown above. Which of the following is a possible graph of $y = f(|x|)$?

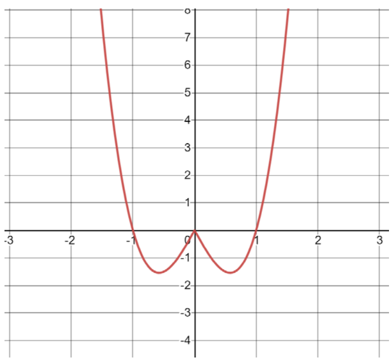
(A)



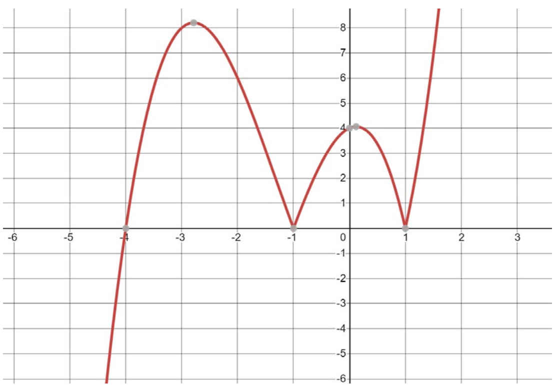
(B)



(C)

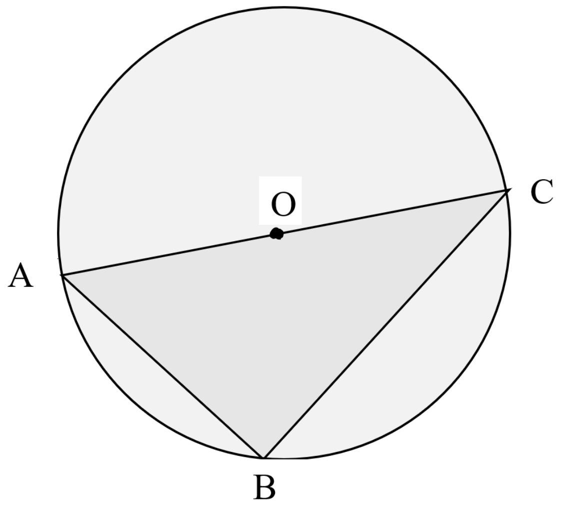


(D)



9. A circle has centre O and radius $\overrightarrow{OA} = \underset{\sim}{a}$

B and C are points on the circle and $\overrightarrow{BC} = \underset{\sim}{b}$



Which one of the following statements must be true?

(A) $\underset{\sim}{a} = \frac{1}{2}\underset{\sim}{b}$

(B) $\underset{\sim}{a} = -\frac{1}{2}\underset{\sim}{b}$

(C) $\underset{\sim}{a} \bullet \underset{\sim}{b} = \underset{\sim}{b} \bullet \underset{\sim}{b}$

(D) $2\underset{\sim}{a} \bullet \underset{\sim}{b} = -\underset{\sim}{b} \bullet \underset{\sim}{b}$

10. X is distributed as a random variable such that $X \sim B(30, 0.4)$. Which of the following pairs of values is true?

(A) $E(X) = 12,\ \mathrm{Var}(X) = \sqrt{7.2}$

(B) $E(X) = 18,\ \sigma^2 = 7.2$

(C) $E(X) = 18,\ \mathrm{Var}(X) = \sqrt{7.2}$

(D) $E(X) = 12,\ \sigma^2 = 7.2$

End of Section I

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour 45 minutes for this section

Answer each question in a SEPARATE writing booklet.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Evaluate **3**

$$\int_0^{\frac{\pi}{6}} \sin^2 2x \, dx$$

(b) Using the substitution $u = \frac{1}{x}$, find the exact value of **3**

$$\int_1^2 \frac{e^{\frac{1}{x}}}{x^2} dx$$

(c) Solve the inequality $\frac{3}{x(2x-1)} > 1$ **3**

(d) Find the coefficient of x in the expansion of $\left(x + \frac{2}{x^2}\right)^{10}$ **3**

(e) Find a vector $\underset{\sim}{d}$ that has a magnitude of 12 and is perpendicular to $\underset{\sim}{c} = -5\underset{\sim}{i} + 2\underset{\sim}{j}$. **3**

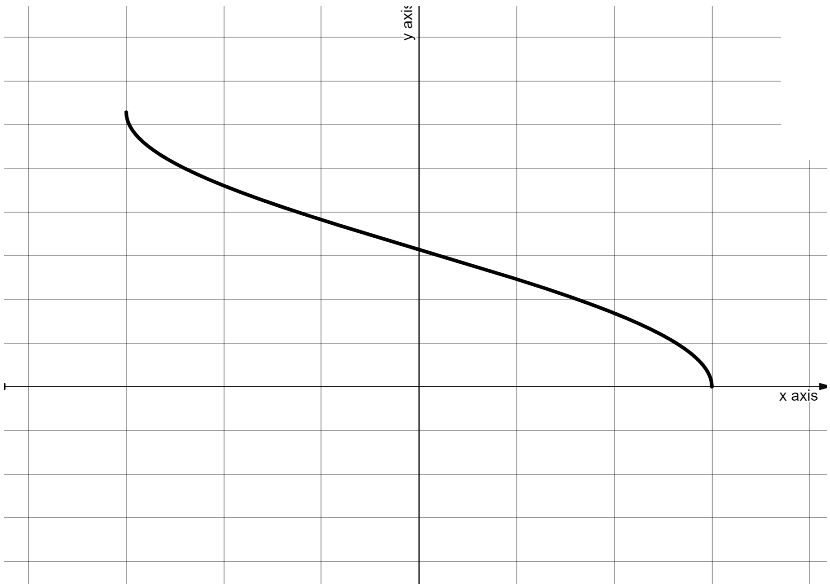
End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) The region enclosed by the curve $y = \dfrac{2}{\sqrt{x-7}}$ and the x-axis between $x = 8$ and $x = 10$ is rotated about the x-axis. Find the exact value of the volume of the solid formed. **2**

(b) Solve $3\sin x + 4\cos x = 2$ for $0 \le x \le 2\pi$ **3**

(c)



The graph above is the function $f(x) = 2\cos^{-1}\dfrac{x}{3}$.

(i) Find the value of the y intercept. **1**

(ii) Find the domain and range of $f(x) = 2\cos^{-1}\dfrac{x}{3}$. **2**

(iii) Calculate the area of the region under the function $f(x) = 2\cos^{-1}\dfrac{x}{3}$ between $x = 0$ and $x = 3$. **2**

Question 12 continues on page 9

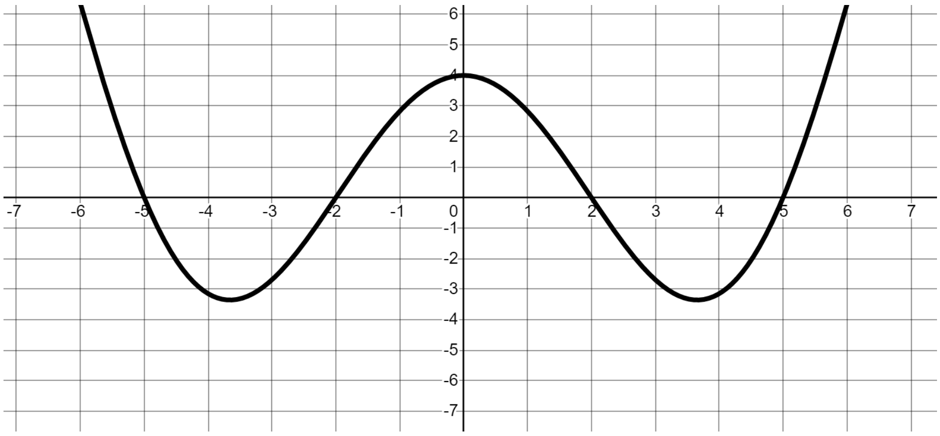
(d) A group of 15 students from a local school is selected for training in soccer to represent the school at grade sport on Wednesday. The probability that a player will not be available to play due to injury or other commitments is 0.14. Find the probability that 3 students will not be available for grade sport on Wednesday, correct to 3 decimal places. **2**

(e) Use the method of mathematical induction to prove that if x is a positive integer, then $(1+x)^n - 1$ is divisible by x for all positive integers , $n \geq 1$ **3**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram shows the graph of the function $y = f(x)$



Draw separate half page sketches of graphs of the following labelling key features

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y = \sqrt{f(x)}$ **2**

(iii) $y = \left(f(x)\right)^2$ **2**

(b) One of the roots of the equation $x^3 + ax^2 + 1 = 0$ is equal to the sum of the other two roots.

(i) Using your knowledge of the sum and products of the roots show that $x = -\dfrac{a}{2}$ is a root of the equation. **1**

(ii) Find the value of a. **3**

Question 13 continues on page 11

(c) A freshly caught fish, initially at 18° C, is placed in a freezer that has a constant unknown temperature of x° C. The cooling rate of the fish is proportional to the difference between the temperature of the freezer and the temperature T° C, of the fish.

It is known that T satisfies the equation $\dfrac{dT}{dt} = -k(T - x)$,

where t is the number of minutes after the fish has been placed in the freezer.

(i) Show that $T = x + Ae^{-kt}$ satisfies this equation where A is a constant. **1**

(ii) If the temperature of the fish is $10^\circ C$ after $7\frac{1}{2}$ minutes, show that the fish's temperature after t minutes is given by $T = x + (18 - x)e^{\frac{2}{15}\log_e\left[\frac{10-x}{18-x}\right]t}$ **3**

(iii) Find the temperature of the fish after 15 minutes when the freezer temperature is $5^\circ C$. Answer to the nearest degree. **1**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) **(i)** Records show that 70% of students at a school participate in athletics. A sample of 83 students at the school is to be taken to determine the proportion who participate in athletics. Show that the mean is 58.1 and standard deviation is 4.17 for such sample proportions. **2**

(ii) Use the table below of values of $P(Z \le z)$, where Z has a Standard Normal distribution, to estimate the probability that a sample of 83 students will contain at most 55 who participate in athletics. **2**

STANDARD NORMAL DISTRIBUTION TABLE

Entries represent $\Pr(Z \le z)$. The value of z to the first decimal is given in the left column. The second decimal is given in the top row.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389

(b) For the following pair of vectors, find the vector projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ **2**

$$\underset{\sim}{a} = -2\underset{\sim}{i} + 3\underset{\sim}{j} \text{ and } \underset{\sim}{b} = 3\underset{\sim}{i} - 5\underset{\sim}{j}.$$

(c) **(i)** Show that $\tan\left(\dfrac{\pi}{4} + A\right) = \dfrac{\cos A + \sin A}{\cos A - \sin A}$ **2**

(ii) Hence show that $\tan\left(\dfrac{\pi}{4} + A\right) = \dfrac{1 + \sin 2A}{\cos 2A}$ **2**

Question 14 continues on page 13

(d) When a particle is fired in the open from a point O at a speed of $40\,\text{ms}^{-1}$ and at an angle of θ above the horizontal, where $0 < \theta < \frac{\pi}{2}$. *You may assume without proof* that the horizontal displacement (x metres) and the vertical displacement (y metres) of the particle from O at time t seconds after firing are given by

$x = 40t\cos\theta$ and $y = 40t\sin\theta - 5t^2$

The particle is fired with the same speed from a point O on the floor of a horizontal tunnel that has a height of 20 metres.

(You may assume the projectile has no size and can reach the maximum height of 20 metres without touching the roof of the tunnel)

(i) Show that the time taken for the projectile to reach the maximum height is $t = 4\sin\theta$ **1**

(ii) Find the angle of projection for this maximum height in the tunnel. **2**

(iii) Show that the exact maximum horizontal range for the projectile in this tunnel is $80\sqrt{3}$ metres. **2**

End of Paper

MC

Q1 $\int_{\frac{3}{\sqrt{2}}}^{3} \frac{4dx}{\sqrt{9-x^2}} = 4\int_{\frac{3}{\sqrt{2}}}^{3} \frac{dx}{\sqrt{9-x^2}}$

$= 4\int_{\frac{3}{\sqrt{2}}}^{3} \frac{dx}{\sqrt{3^2-x^2}}$

$= 4\left[\sin^{-1}\left(\frac{x}{3}\right)\right]_{\frac{3}{\sqrt{2}}}^{3}$

$= 4\left[\sin^{-1}\left(\frac{3}{3}\right) - \sin^{-1}\left(\frac{3/\sqrt{2}}{3}\right)\right]$ $\qquad \frac{3}{\sqrt{2}} \div 3 = \frac{\cancel{3}}{\sqrt{2}} \times \frac{1}{\cancel{3}} = \frac{1}{\sqrt{2}}$

$= 4\left[\sin^{-1}(1) - \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right]$

$= 4\left[\frac{\pi}{2} - \frac{\pi}{4}\right]$

$= 4\left[\frac{2\pi-\pi}{4}\right]$

$= \pi$ (D)

Q2 $A = \pi r^2$ $\qquad \frac{dA}{dr} = 10m^2/s.$

$\frac{dA}{dr} = 2\pi r$

$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}.$ $\qquad$ A

$10 = 2\pi r \times \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{5}{\pi r}$

Q3 $x^3 - 2x^2 - 4x + 8 = 0$ $\quad \alpha, \beta, \gamma$.

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$$

$\alpha\beta + \beta\gamma + \alpha\gamma = -4$

$\alpha\beta\gamma = -8$.

$$= \frac{-4}{-8}$$

$$= \frac{1}{2} \quad \text{(D)}$$

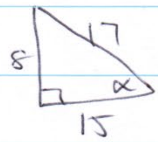
Q4 $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$$= \frac{8}{17} \times \frac{3}{5} + \frac{15}{17} \times \frac{4}{5}$$

$$= \frac{24}{85} + \frac{12}{17}$$

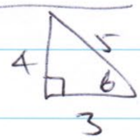
$$= \frac{84}{85} \quad \text{(B)}$$

$\sin\alpha = \frac{8}{17}$



$\cos\alpha = \frac{15}{17}$

$\sin\beta = \frac{4}{5}$



$\cos\beta = \frac{3}{5}$.

Q5 $y = \cos^{-1}\left(\frac{1}{x}\right)$

Let $u = \frac{1}{x}$, $\frac{du}{dx} = -x^{-2}$
$= x^{-1}$ $\qquad = \frac{-1}{x^2}$

And $y = \cos^{-1} u$, $\frac{dy}{du} = -\frac{1}{\sqrt{1-u^2}} = -\frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}}$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{-1}{\sqrt{1-\frac{1}{x^2}}} \times \frac{-1}{x^2}$$

PTO

$$\frac{dy}{dx} = -\frac{1}{\sqrt{\frac{x^2-1}{x^2}}} \times \frac{-1}{x^2}$$

$$= \frac{1}{x^2\sqrt{\frac{x^2-1}{x^2}}}$$

$$= \frac{1}{\frac{x^2}{x}\sqrt{x^2-1}}$$

$$= \frac{1}{x\sqrt{x^2-1}} \quad \underline{\underline{(B)}}$$

<u>Q6</u> $\int \sec^2\theta \tan^2\theta \, d\theta$

$u = \tan\theta$

$\frac{du}{d\theta} = \sec^2\theta$.

$du = \sec^2\theta \, d\theta$.

$$= \int u^2 \, du$$

$$= \frac{u^3}{3} + c.$$

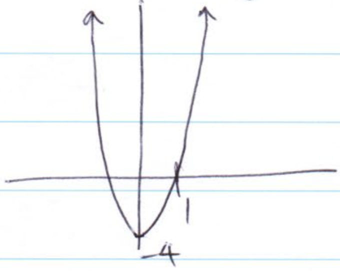
$$= \frac{1}{3}\tan^3\theta + c. \quad \underline{\underline{(B)}}$$

<u>Q7</u> At least 1 boy = Total − No boys.

$= {}^{10}C_5 - {}^{6}C_5$

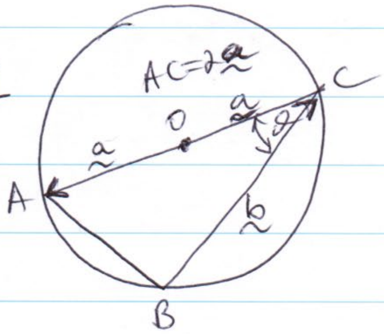
$= \binom{10}{5} - 6 \quad \underline{\underline{(D)}}$

⑧ $y=f|x|$, the graph on RHS of y-axis is reflected about the y-axis.



(A)

Q9



Your diagram looks stretched ..., not a circle.

Use $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}| \times |\underset{\sim}{b}| \cos\theta$.

$2\underset{\sim}{a} \cdot \underset{\sim}{b} = 2|\underset{\sim}{a}| \times |\underset{\sim}{b}| \cos\theta$.

$$\cos\theta = \frac{2\underset{\sim}{a} \cdot \underset{\sim}{b}}{2|\underset{\sim}{a}| \times |\underset{\sim}{b}|}$$

$$= \frac{2|\underset{\sim}{a}||\underset{\sim}{b}| \times |\underset{\sim}{b}|}{2|\underset{\sim}{a}|}$$

$$= |b|^2$$

$$= \underset{\sim}{b} \cdot \underset{\sim}{b}$$

And $\cos\theta = \dfrac{\underset{\sim}{b}}{2\underset{\sim}{a}}$

$$= \frac{|b|}{2|\underset{\sim}{a}|}$$

$\therefore 2\underset{\sim}{a} \cdot \underset{\sim}{b} - \underset{\sim}{b} \cdot \underset{\sim}{b}$

(D)

Caringbah High School

Examination:....................................

Student Number

Q10 $X \sim B(30, 0.4)$

$n = 30, \; p = 0.4, \; q = 0.6$

$E(X) = 30 \times 0.4 = np$

$= 12.$

$\sigma^2 = npq = np(1-p)$

$= 30 \times 0.4 \times 0.6$

$= 7.2.$

$\therefore E(X) = 12, \; \sigma^2 = 7.2$ (D)

Caringbah High School

Student Number

Examination:..................................

Q11

(a) $\int_0^{\pi/6} \sin^2 2x \, dx = \int_0^{\pi/6} \left(\frac{1}{2} - \frac{1}{2}\cos 4x\right) dx$.

$= \left[\frac{1}{2}x - \frac{1}{8}\sin 4x\right]_0^{\pi/6}$

$= \left(\frac{1}{2}\left(\frac{\pi}{6}\right) - \frac{1}{8}\sin 4\left(\frac{\pi}{6}\right) - 0\right)$

$= \frac{\pi}{12} - \frac{1}{8}\sin\frac{2\pi}{3} - 0$

$= \frac{\pi}{12} - \frac{1}{8} \times \frac{\sqrt{3}}{2}$

$= \frac{\pi}{12} - \frac{\sqrt{3}}{16}$.

(b) $\int_1^2 \frac{e^{\frac{1}{x}}}{x^2} dx$

$u = \frac{1}{x} = x^{-1}$

$\frac{du}{dx} = -x^{-2} = \frac{-1}{x^2}$

$du = -\frac{1}{x^2} dx$.

when $x=2$, $u = \frac{1}{2}$

" $x=1$, $u=1$

$= \int_1^{1/2} e^u du$

$= \left[e^u\right]_1^{1/2}$

$= e^{1/2} - e$

Q11 (cont)

(c) $\dfrac{3}{x(2x-1)} > 1$, $x \neq 0, \frac{1}{2}$

$3 > x(2x-1)$

$x(2x-1) < 3$.

$2x^2 - x - 3 < 0$.

$(x+1)(2x-3) = 0$

$x = -1, \frac{3}{2}$.

Test: ✗ ✓ ✗ ✓ ✗

$-1 \quad 0 \quad \frac{1}{2} \quad \frac{3}{2}$

$\therefore -1 < x < 0$ & $\frac{1}{2} < x < \frac{3}{2}$

(d) $\left(x + \dfrac{2}{x^2}\right)^{10}$

$T_{k+1} = {}^nC_k \cdot a^{n-k} \cdot b^k$

$= {}^{10}C_k \cdot x^{10-k} \cdot (2x^{-2})^k$

$= {}^{10}C_k \, x^{10-k} \cdot 2^k \cdot x^{-2k}$

$= {}^{10}C_k \cdot 2^k \cdot x^{10-3k}$.

To find coefficient of x: $10 - 3k = 1$

$3k = 9$

$k = 3$.

$\therefore$ coefficient of x is ${}^{10}C_3 \cdot 2^3 = 960$.

Q11 (cont)

(e) $\underset{\sim}{c} = -5\underset{\sim}{i} + 2\underset{\sim}{j}$ magnitude = 12.

$\underset{\sim}{c} \times \underset{\sim}{d} = 0$ (1)

$$\begin{bmatrix} -5 \\ 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0.$$

$-5x + 2y = 0.$

$2y = 5x$

$y = \frac{5}{2}x$ ... (2)

$x^2 + y^2 = 144$

$x^2 + \left(\frac{5}{2}\right)^2 x^2 = 144.$

$x^2 + \frac{25}{4}x^2 = 144.$

$\frac{29x^2}{4} = 144$

$x^2 = \frac{576}{29}$

$x = \frac{24}{\sqrt{29}}$ ... (1)

sub (1) into (2). $y = \frac{5}{2}\left(\frac{24}{\sqrt{29}}\right)$

$= \frac{60}{\sqrt{29}}.$

$\therefore \underset{\sim}{d} = \left(\frac{24}{\sqrt{29}} i + \frac{60}{\sqrt{29}} j\right)$

$= \frac{12}{\sqrt{29}}\left(2\underset{\sim}{i} + 5\underset{\sim}{j}\right)$

Q12

(a) $y = \dfrac{2}{\sqrt{x-7}}$, $x=8, x=10$.

$V = \pi \int_a^b y^2\,dx$. $y^2 = \dfrac{4}{x-7}$

$= \pi \int_8^{10} \dfrac{4}{x-7}\,dx$.

$= 4\pi \int_8^{10} \dfrac{1}{x-7}\,dx$.

$= 4\pi \left[\ln(x-7)\right]_8^{10}$

$= 4\pi \left[\ln(10-7) - \ln(8-7)\right]$

$= 4\pi \left[\ln(3) - \ln(1)\right]$

$= 4\pi \left[\ln(3) - 0\right]$

$= 4\pi \ln(3)$

Your answer is different to mine... I think you subbed $x=8$ incorrectly. AN.

(b) $3\sin x + 4\cos x = 2$ $0° \le x \le 360°$

$R\sin(x+\alpha) = R(\sin x\cos\alpha + \cos x\sin\alpha)$

$= R\sin x\cos\alpha + R\cos x\sin\alpha$.

$\therefore R\cos\alpha = 3$, $R\sin\alpha = 4$. $R^2 = 3^2 + 4^2$

$5\cos\alpha = 3$ $5\sin\alpha = 4$ $= 25$

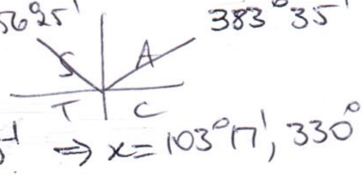
$\cos\alpha = \frac{3}{5}$ $\sin\alpha = \frac{4}{5}$ $\therefore R = 5$.

$\alpha = \cos^{-1}\left(\frac{3}{5}\right)$ $\alpha = \sin^{-1}\left(\frac{4}{5}\right)$

Now $5\sin\left(x + \cos^{-1}\left(\frac{3}{5}\right)\right) = 2$.

$5\sin(x + 53°8') = 2$, $53°8' \le x + 53°8' \le 413°8'$

$\sin(x+53°8') = \frac{2}{5}$



$x + 53°8' = \sin^{-1}\left(\frac{2}{5}\right)$

$x + 53°8' = 156°25', 383°35' \Rightarrow x = 103°17', 330°27'$

(9)

Q12 (c) Q12 cont

$f(x) = 2\cos^{-1}\left(\frac{x}{3}\right)$

(i) y-int, $x=0$.

$y = 2\cos^{-1}(0)$

$= 2 \times \frac{\pi}{2}$

$\therefore y = \pi$

(ii) Domain: $-1 \le \frac{x}{3} \le 1$, Range: $0 \le \frac{y}{2} \le \pi$

$\therefore -3 \le x \le 3$ $\qquad$ $0 \le y \le 2\pi$.

(iii) $A = \int_0^{\pi} 3\cos\frac{y}{2}\,dy$

$= 3\int_0^{\pi} \cos\frac{y}{2}\,dy$

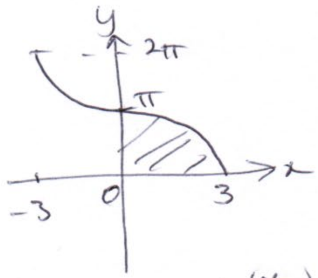
$= 6\left[\sin\frac{y}{2}\right]_0^{\pi}$

$= 6\left(\sin\frac{\pi}{2} - \sin 0\right)$

$= 6\sin\frac{\pi}{2}$

$= 6(1)$

$= 6\,u^2$



$y = 2\cos^{-1}\left(\frac{x}{3}\right)$

$\frac{y}{2} = \cos^{-1}\left(\frac{x}{3}\right)$

$\frac{x}{3} = \cos\frac{y}{2}$

$x = 3\cos\frac{y}{2}$.

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Q12 continued

d) $P = {}^nC_r\, p^n q^{n-r}$

$= {}^{15}C_3\,(0.14)^3\,(0.86)^{15-3}$

$= {}^{15}C_3\,(0.14)^3(0.86)^{12}$

$= 0.2043510712\ldots$

$= 0.204$ (3 dp)

e) $(1+x)^n - 1$ is divisible by x, $n \geq 1$

Let $n=1$, $1+x^1-1 = x$

$\therefore$ divisible by x.

$\therefore n=1$ is true.

Assume $n=k$, $(1+x)^k - 1 = Mx$, where M is any integer.

$(1+x)^k = Mx+1$

Prove $n=k+1$

$(1+x)^{k+1} - 1 = (1+x)^k(1+x) - 1$

$= (Mx+1)(1+x) - 1$

$= Mx + Mx^2 + 1 + x - 1$

$= Mx + Mx^2 + x$

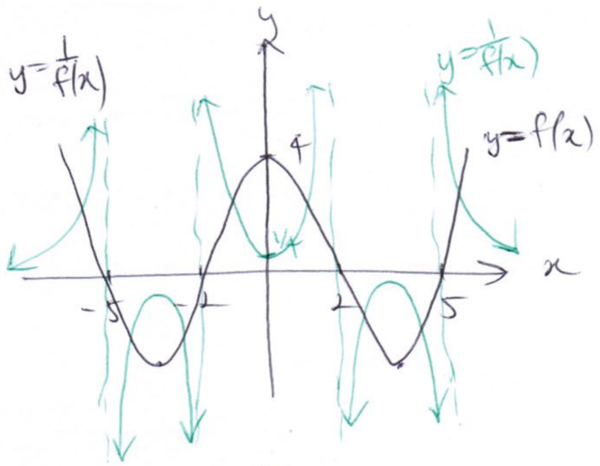
$= x(M + Mx + 1)$

which is divisible by x.

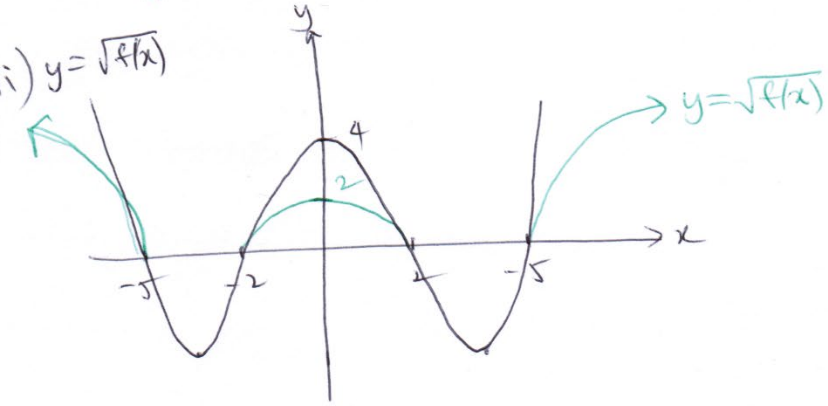
$\therefore$ true by MI for all $n \geq 1$.

Q13

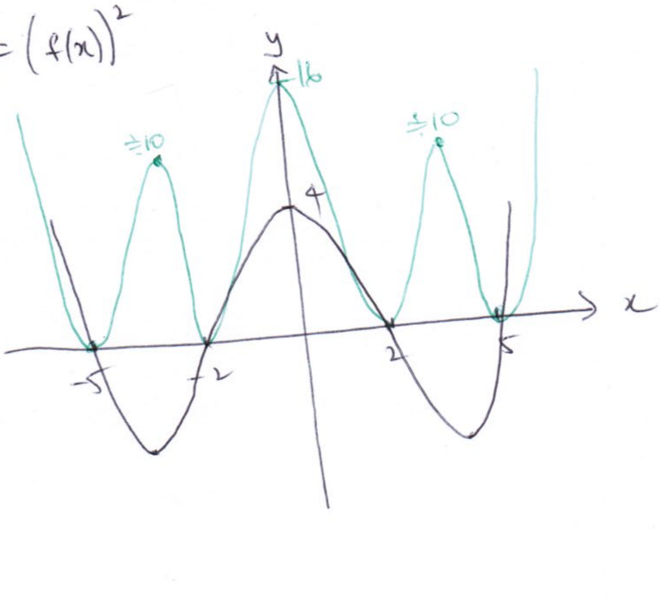
(a) (i) $y=\frac{1}{f(x)}$



(ii) $y=\sqrt{f(x)}$



(iii) $y=(f(x))^2$



Q13 (cont)

(b) (i) $x^3+ax^2+1=0$

$\alpha = \beta+\gamma$.

$\alpha+\beta+\gamma = -\frac{b}{a}$

$\therefore \alpha+\beta+\gamma = -a$

$\alpha+\alpha=-a$

$2\alpha=-a$

$\alpha = -\frac{a}{2}$

$\therefore x = -\frac{a}{2}$.

(ii) $\alpha\beta\gamma = -\frac{d}{a}$

$\alpha\beta\gamma = -1$ & $\beta\gamma = \frac{-1}{\alpha} = \frac{2}{a}$.

Now, $\alpha\beta+\beta\gamma+\alpha\gamma=0$

$\alpha\beta + \frac{2}{a} + \alpha\gamma = 0$.

$\alpha(\beta+\gamma) + \frac{2}{a} = 0$.

$-\frac{a}{2}\left(-\frac{a}{2}\right) + \frac{2}{a} = 0$.

You have a minus & I have a plus... Please check!!!

$\frac{a^2}{4} + \frac{2}{a} = 0$.

$\frac{a^2}{4} = -\frac{2}{a}$

$a^3 = -8$

$a = -2$

Q13 (cont)

(c)(i) $T = x + Ae^{-kt}$

$Ae^{-kt} = T - x$.

$\frac{dT}{dt} = -ke^{-kt}$

$= -k(T - x)$

$\therefore \frac{dT}{dt} = -k(T - x)$

(ii) $T = x + Ae^{-kt}$

when $t = 0$, $T = 18$

$18 = x + Ae^{0}$

$18 = x + A$

$\therefore A = 18 - x$.

ie $T = x + (18 - x)e^{-kt}$

when $t = 7.5$, $T = 10$,

$10 = x + (18 - x)e^{-k(7.5)}$

$10 - x = (18 - x)e^{-7.5k}$

$\frac{10 - x}{18 - x} = e^{-7.5k}$.

$\frac{10 - x}{18 - x} = e^{-\frac{15}{2}k}$.

$\ln\left(\frac{10 - x}{18 - x}\right) = \ln e^{-\frac{15}{2}k}$.

$-\frac{15}{2}k = \ln\left(\frac{10 - x}{18 - x}\right)$

$k = -\frac{2}{15}\ln\left(\frac{10 - x}{18 - x}\right)$

$\therefore T = x + (18 - x)e^{-\left(-\frac{2}{15}\ln\left(\frac{10-x}{18-x}\right)\right)t}$

$\therefore T = x + (18 - x)e^{\frac{2}{15}\ln\left(\frac{10-x}{18-x}\right)t}$

(iii) when $t = 15$, $x = 5$

$T = 5 + (18 - 5)e^{\frac{2}{15}\ln\left(\frac{10-5}{18-5}\right)(15)}$

$= 5 + 13e^{\frac{2}{15}\ln\left(\frac{5}{13}\right)(15)}$

$= 5 + 13e^{2\ln(5/13)}$

$= 6.923\ldots$ minutes.

$\doteqdot 7$ minutes.

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Q14

(a) (i) $n=96,\ p=0.6,\ q=0.4$

$\mu=np$ $\qquad\sigma=\sqrt{npq}$

$=96\times0.6$ $\qquad=\sqrt{96\times0.6\times0.4}$

$=57.6$ $\qquad=\sqrt{23.04}$

$\qquad=4.8$

(ii) $z=\dfrac{54-57.6}{4.8}=-0.75$

$P(z\leq0.75)=0.7734$

$\therefore P(z>0.75)=1-0.7734$

$=0.2266$

(b) $\underset{\sim}{a}=-2\underset{\sim}{i}+3\underset{\sim}{j}$, $\underset{\sim}{b}=3\underset{\sim}{i}-5\underset{\sim}{j}$

$\text{proj}_{\underset{\sim}{b}}\underset{\sim}{a}=\dfrac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\underset{\sim}{b}\cdot\underset{\sim}{b}}\times\underset{\sim}{b}$ $\qquad\underset{\sim}{a}=\begin{bmatrix}-2\\3\end{bmatrix},\ \underset{\sim}{b}=\begin{bmatrix}3\\5\end{bmatrix}$

$\underset{\sim}{a}\cdot\underset{\sim}{b}=-6-15=-21$

$=\dfrac{-21}{34}(3\underset{\sim}{i}-5\underset{\sim}{j})$ $\qquad\underset{\sim}{b}\cdot\underset{\sim}{b}=34$

15

Q14 (cont)

(c) (i) $\tan\left(\frac{\pi}{4}+A\right) = \frac{\cos A + \sin A}{\cos A - \sin A}$

LHS:

$\tan\left(\frac{\pi}{4}+A\right) = \frac{\tan\frac{\pi}{4} + \tan A}{1 - \tan\frac{\pi}{4}\tan A}$

$= \frac{1+\tan A}{1-\tan A}$

$= \frac{1+\frac{\sin A}{\cos A}}{1-\frac{\sin A}{\cos A}}$

$= \frac{\frac{\cos A + \sin A}{\cos A}}{\frac{\cos A - \sin A}{\cos A}}$

$= \frac{\cos A + \sin A}{\cos A - \sin A}$

$=$ RHS.

$\therefore$ LHS $=$ RHS.

Q14(cont)

(c)(ii) $\tan\left(\frac{\pi}{4}+A\right) = \frac{1+\sin 2A}{\cos 2A}$

Using $\tan\left(\frac{\pi}{4}+A\right) = \frac{\cos A+\sin A}{\cos A-\sin A}$ (from (i))

Then: $\overset{\text{LHS:}}{\frac{\cos A+\sin A}{\cos A-\sin A}} = \frac{\cos A+\sin A}{\cos A-\sin A} \times \frac{\cos A+\sin A}{\cos A+\sin A}$

$= \frac{\cos^2 A + 2\sin A\cos A + \sin^2 A}{\cos^2 A - \sin^2 A}$

$= \frac{1+2\sin A\cos A}{\cos^2 A-\sin^2 A}$

$= \frac{1+\sin 2A}{\cos 2A}$

$=$ RHS.

$\therefore$ LHS $=$ RHS.

Q14 (cont)

(d)(i) $y = 40t\sin\theta - 5t^2$

$\dot{y} = 40\sin\theta - 10t$

For max height $\dot{y} = 0$.

$40\sin\theta - 10t = 0$

$10t = 40\sin\theta$

$t = \frac{40\sin\theta}{10}$

$t = 4\sin\theta$.

(ii) max height $y = 20$.

$y = 40t\sin\theta - 5t^2$

$20 = 40(4\sin\theta)\sin\theta - 5(4\sin\theta)^2$

$20 = 160\sin^2\theta - 80\sin^2\theta$

$20 = 80\sin^2\theta$.

$\sin^2\theta = \frac{1}{4}$

$\sin\theta = \pm\frac{1}{2}$.

$\theta = 30°$ OR $\theta = \frac{\pi}{6}$.

(iii) Max range: $y = 0$ & at $2 \times$ max height. ($t = 8\sin\theta$)

$x = 40t\cos\theta$

$x = 40(8\sin\theta)\cos\theta$.

$x = 320\sin\theta\cos\theta$.

$= 160(2\sin\theta\cos\theta)$

$= 160\sin 2\theta$.

$= 160\sin 2(\pi/6)$

$x = 160\sin\frac{\pi}{3}$

$= 160 \times \frac{\sqrt{3}}{2}$

$= 80\sqrt{3}$ m

THE END!!!